MINIMAL SUBMANIFOLDS

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1. INTRODUCTION

Soap films, soap bubbles, and surface tension were extensively studied by the Belgian physicist and inventor (the inventor of the stroboscope) Joseph Plateau in the first half of the nineteenth century. At least since his studies, it has been known

that the right mathematical model for soap films are minimal surfaces – the soap film is in a state of minimum energy when it is covering the least possible amount of area. Minimal surfaces and equations like the minimal surface equation have served as mathematical models for many physical problems.

The field of minimal surfaces dates back to the publication in 1762 of Lagrange's famous memoir "Essai d'une nouvelle méthode pour déterminer les maxima et les minima des formules intégrales indéfinies". Euler had already in a paper published in 1744 discussed minimizing properties of the surface now known as the catenoid, but he only considered variations within a certain class of surfaces. In the almost one quarter of a millennium that has past since Lagrange's memoir minimal surfaces has remained a vibrant area of research and there are many reasons why. The study of minimal surfaces was the birthplace of regularity theory. It lies on the intersection of nonlinear elliptic PDE, geometry, topology and general relativity.

In what follows we give a quick tour through many of the classical results in the field of minimal submanifolds, starting at the definition.

The field of minimal surfaces remain extremely active and has very recently seen major developments that have solved many longstanding open problems and conjectures; for more on this, see the expanded version of this survey [CM3]. See also the recent surveys [MeP], [Pz] and the expository article [CM4].

Throughout this survey, we refer to [CM1] for references unless otherwise noted.

Part 1. **Classical and almost classical results**

Let $\Sigma \subset \mathbf{R}^n$ be a <u>smooth</u> k-dimensional submanifold (possibly with boundary) and $C_0^\infty(N\Sigma)$ the space of all infinitely differentiable, compactly supported, normal vector fields on Σ. Given Φ in $C_0^\infty(N\Sigma)$, consider the one–parameter variation

$$(1.1) \qquad \Sigma_{t,\Phi} = \{x + t\,\Phi(x)|x \in \Sigma\}\,.$$

The so called first variation formula of volume is the equation (integration is with respect to $d$vol)

$$(1.2) \qquad \frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(\Sigma_{t,\Phi}) = \int_\Sigma \langle \Phi\,,\,\mathbf{H}\rangle\,,$$

where $\mathbf{H}$ is the mean curvature (vector) of Σ. (When Σ is noncompact, then $\Sigma_{t,\Phi}$ in (1.2) is replaced by $\Gamma_{t,\Phi}$, where Γ is any compact set containing the support of Φ.) The submanifold Σ is said to be a *minimal* submanifold (or just minimal) if

$$(1.3) \qquad \frac{d}{dt}\bigg|_{t=0} \mathrm{Vol}(\Sigma_{t,\Phi}) = 0 \qquad \text{for all } \Phi \in C_0^\infty(N\Sigma)$$

or, equivalently by (1.2), if the mean curvature $\mathbf{H}$ is identically zero. Thus Σ is minimal if and only if it is a critical point for the volume functional. (Since a critical point is not necessarily a minimum the term "minimal" is misleading, but it is time honored. The equation for a critical point is also sometimes called the Euler–Lagrange equation.)

Suppose now, for simplicity, that Σ is an oriented hypersurface with unit normal $\mathbf{n}_\Sigma$. We can then write a normal vector field $\Phi \in C_0^\infty(N\Sigma)$ as $\Phi = \phi\mathbf{n}_\Sigma$, where function ϕ is in the space $C_0^\infty(\Sigma)$ of infinitely differentiable, compactly supported

functions on Σ. Using this, a computation shows that if Σ is minimal, then

$$(1.4) \qquad \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(\Sigma_{t,\phi\mathbf{n}_\Sigma}) = -\int_\Sigma \phi\, L_\Sigma \phi\,,$$

where

$$(1.5) \qquad L_\Sigma \phi = \Delta_\Sigma \phi + |A|^2 \phi$$

is the second variational (or Jacobi) operator. Here Δ_Σ is the Laplacian on Σ and A is the second fundamental form. So $|A|^2 = \kappa_1^2 + \kappa_2^2 + \cdots + \kappa_{n-1}^2$, where $\kappa_1, \ldots \kappa_{n-1}$ are the principal curvatures of Σ and $\mathbf{H} = (\kappa_1 + \cdots + \kappa_{n-1})\,\mathbf{n}_\Sigma$. A minimal submanifold Σ is said to be stable if

$$(1.6) \qquad \frac{d^2}{dt^2}\bigg|_{t=0} \mathrm{Vol}(\Sigma_{t,\Phi}) \geq 0 \qquad \text{for all } \Phi \in C_0^\infty(N\Sigma)\,.$$

Integrating by parts in (1.4), we see that stability is equivalent to the so called stability inequality

$$(1.7) \qquad \int |A|^2\, \phi^2 \leq \int |\nabla\phi|^2\,.$$

More generally, the *Morse index* of a minimal submanifold is defined to be the number of negative eigenvalues of the operator L. Thus, a stable submanifold has Morse index zero.

1.1. The Gauss map.
Let $\Sigma^2 \subset \mathbf{R}^3$ be a surface (not necessarily minimal). The *Gauss map* is a continuous choice of a unit normal $\mathbf{n} : \Sigma \to \mathbf{S}^2 \subset \mathbf{R}^3$. Observe that there are two choices of such a map $\mathbf{n}$ and $-\mathbf{n}$ corresponding to a choice of orientation of Σ. If Σ is minimal, then the Gauss map is an (anti) conformal map since the eigenvalues of the Weingarten map are κ_1 and $\kappa_2 = -\kappa_1$. Moreover, for a minimal surface

$$(1.8) \qquad |A|^2 = \kappa_1^2 + \kappa_2^2 = -2\,\kappa_1\,\kappa_2 = -2\,K_\Sigma\,,$$

where K_Σ is the Gauss curvature. It follows that the area of the Gauss map is a multiple of the total curvature.

1.2. Minimal graphs.
Suppose that $u : \Omega \subset \mathbf{R}^2 \to \mathbf{R}$ is a C^2 function. The graph of u

$$(1.9) \qquad \mathrm{Graph}_u = \{(x, y, u(x, y)) \mid (x, y) \in \Omega\}\,.$$

has area

$$(1.10) \qquad \mathrm{Area}(\mathrm{Graph}_u) = \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)|$$
$$= \int_\Omega \sqrt{1 + u_x^2 + u_y^2} = \int_\Omega \sqrt{1 + |\nabla u|^2}\,,$$

and the (upward pointing) unit normal is

$$(1.11) \qquad \mathbf{n} = \frac{(1, 0, u_x) \times (0, 1, u_y)}{|(1, 0, u_x) \times (0, 1, u_y)|} = \frac{(-u_x, -u_y, 1)}{\sqrt{1 + |\nabla u|^2}}\,.$$

Therefore for the graphs $\mathrm{Graph}_{u+t\eta}$ where $\eta|\partial\Omega = 0$ we get that

$$(1.12) \qquad \mathrm{Area}(\mathrm{Graph}_{u+t\eta}) = \int_\Omega \sqrt{1 + |\nabla u + t\,\nabla\eta|^2}$$

hence

$$(1.13) \qquad \frac{d}{dt}\Big|_{t=0} \mathrm{Area}(\mathrm{Graph}_{u+t\eta}) = \int_\Omega \frac{\langle \nabla u, \nabla \eta \rangle}{\sqrt{1+|\nabla u|^2}}$$

$$= -\int_\Omega \eta \,\mathrm{div}\,\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right).$$

It follows that the graph of u is a critical point for the area functional if and only if u satisfies the divergence form equation

$$(1.14) \qquad \mathrm{div}\,\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) = 0.$$

Next we want to show that the graph of a function on Ω satisfying the minimal surface equation, i.e., satisfying (1.14), is not just a critical point for the area functional but is actually area-minimizing amongst surfaces in the cylinder $\Omega \times \mathbf{R} \subset \mathbf{R}^3$. To show this, extend first the unit normal $\mathbf{n}$ of the graph in (1.11) to a vector field, still denoted by $\mathbf{n}$, on the entire cylinder $\Omega \times \mathbf{R}$. Let ω be the two-form on $\Omega \times \mathbf{R}$ given by that for $X, Y \in \mathbf{R}^3$

$$(1.15) \qquad \omega(X,Y) = \det(X,Y,\mathbf{n}).$$

An easy calculation shows that

$$(1.16) \qquad d\omega = \frac{\partial}{\partial x}\left(\frac{-u_x}{\sqrt{1+|\nabla u|^2}} \right) + \frac{\partial}{\partial y}\left(\frac{-u_y}{\sqrt{1+|\nabla u|^2}} \right) = 0,$$

since u satisfies the minimal surface equation. In sum, the form ω is closed and, given any X and Y at a point (x,y,z),

$$(1.17) \qquad |\omega(X,Y)| \leq |X \times Y|,$$

where equality holds if and only if

$$(1.18) \qquad X,Y \subset T_{(x,y,u(x,y))}\,\mathrm{Graph}_u\,.$$

Such a form ω is called a *calibration*. From this, we have that if $\Sigma \subset \Omega \times \mathbf{R}$ is any other surface with $\partial\Sigma = \partial\,\mathrm{Graph}_u$, then by Stokes' theorem since ω is closed,

$$(1.19) \qquad \mathrm{Area}(\mathrm{Graph}_u) = \int_{\mathrm{Graph}_u} \omega = \int_\Sigma \omega \leq \mathrm{Area}(\Sigma).$$

This shows that Graph_u is area-minimizing among all surfaces in the cylinder and with the same boundary. If the domain Ω is convex, the minimal graph is absolutely area-minimizing. To see this, observe first that if Ω is convex, then so is $\Omega \times \mathbf{R}$ and hence the nearest point projection $P : \mathbf{R}^3 \to \Omega \times \mathbf{R}$ is a distance nonincreasing Lipschitz map that is equal to the identity on $\Omega \times \mathbf{R}$. If $\Sigma \subset \mathbf{R}^3$ is any other surface with $\partial\Sigma = \partial\,\mathrm{Graph}_u$, then $\Sigma' = P(\Sigma)$ has $\mathrm{Area}(\Sigma') \leq \mathrm{Area}(\Sigma)$. Applying (1.19) to Σ', we see that $\mathrm{Area}(\mathrm{Graph}_u) \leq \mathrm{Area}(\Sigma')$ and the claim follows.

If $\Omega \subset \mathbf{R}^2$ contains a ball of radius r, then, since $\partial B_r \cap \mathrm{Graph}_u$ divides ∂B_r into two components at least one of which has area at most equal to $(\mathrm{Area}(\mathbf{S}^2)/2)\,r^2$, we get from (1.19) the crude estimate

$$(1.20) \qquad \mathrm{Area}(B_r \cap \mathrm{Graph}_u) \leq \frac{\mathrm{Area}(\mathbf{S}^2)}{2}\,r^2.$$

When the domain Ω is convex, it is not hard to see that the minimal graph is absolutely area-minimizing.

Very similar calculations to the ones above show that if $\Omega \subset \mathbf{R}^{n-1}$ and $u : \Omega \to \mathbf{R}$ is a C^2 function, then the graph of u is a critical point for the area functional if and only if u satisfies (1.14). Moreover, as in (1.19), the graph of u is actually area-minimizing. Consequently, as in (1.20), if Ω contains a ball of radius r, then

$$(1.21) \qquad \mathrm{Vol}(B_r \cap \mathrm{Graph}_u) \leq \frac{\mathrm{Vol}(\mathbf{S}^{n-1})}{2} r^{n-1} .$$

1.3. **The maximum principle.** The first variation formula, (1.2), showed that a smooth submanifold is a critical point for area if and only if the mean curvature vanishes. We will next derive the weak form of the first variation formula which is the basic tool for working with "weak solutions" (typically, stationary varifolds). Let X be a vector field on $\mathbf{R}^n$. We can write the divergence $\mathrm{div}\,_\Sigma X$ of X on Σ as

$$(1.22) \qquad \mathrm{div}\,_\Sigma X = \mathrm{div}\,_\Sigma X^T + \mathrm{div}\,_\Sigma X^N = \mathrm{div}\,_\Sigma X^T + \langle X, \mathbf{H} \rangle ,$$

where X^T and X^N are the tangential and normal projections of X. In particular, we get that, for a minimal submanifold,

$$(1.23) \qquad \mathrm{div}\,_\Sigma X = \mathrm{div}\,_\Sigma X^T .$$

Moreover, from (1.22) and Stokes' theorem, we see that Σ is minimal if and only if for all vector fields X with compact support and vanishing on the boundary of Σ,

$$(1.24) \qquad \int_\Sigma \mathrm{div}\,_\Sigma X = 0 .$$

The key point is that (1.24) makes sense as long as we can define the divergence on Σ. As a consequence of (1.24), we will show the following proposition:

Proposition 1.1. $\Sigma^k \subset \mathbf{R}^n$ *is minimal if and only if the restrictions of the coordinate functions of $\mathbf{R}^n$ to Σ are harmonic functions.*

Proof. Let η be a smooth function on Σ with compact support and $\eta|\partial\Sigma = 0$, then

$$(1.25) \qquad \int_\Sigma \langle \nabla_\Sigma \eta, \nabla_\Sigma x_i \rangle = \int_\Sigma \langle \nabla_\Sigma \eta, e_i \rangle = \int_\Sigma \mathrm{div}\,_\Sigma (\eta\, e_i) .$$

From this, the claim follows easily. $\qquad\square$

Recall that if $\Xi \subset \mathbf{R}^n$ is a compact subset, then the smallest convex set containing Ξ (the convex hull, $\mathrm{Conv}(\Xi)$) is the intersection of all half–spaces containing Ξ. The maximum principle forces a compact minimal submanifold to lie in the convex hull of its boundary (this is the "convex hull property"):

Proposition 1.2. *If $\Sigma^k \subset \mathbf{R}^n$ is a compact minimal submanifold, then $\Sigma \subset \mathrm{Conv}(\partial\Sigma)$.*

Proof. A half–space $H \subset \mathbf{R}^n$ can be written as

$$(1.26) \qquad H = \{ x \in \mathbf{R}^n \,|\, \langle x, e \rangle \leq a \} ,$$

for a vector $e \in \mathbf{S}^{n-1}$ and constant $a \in \mathbf{R}$. By Proposition 1.1, the function $u(x) = \langle e, x \rangle$ is harmonic on Σ and hence attains its maximum on $\partial\Sigma$ by the maximum principle. $\qquad\square$

Another application of (1.23), with a different choice of vector field X, gives that for a k-dimensional minimal submanifold Σ

$$(1.27) \qquad \Delta_\Sigma |x - x_0|^2 = 2\operatorname{div}_\Sigma (x - x_0) = 2k\,.$$

Later we will see that this formula plays a crucial role in the monotonicity formula for minimal submanifolds.

The argument in the proof of the convex hull property can be rephrased as saying that as we translate a hyperplane towards a minimal surface, the first point of contact must be on the boundary. When Σ is a hypersurface, this is a special case of the strong maximum principle for minimal surfaces:

Lemma 1.3. *Let $\Omega \subset \mathbf{R}^{n-1}$ be an open connected neighborhood of the origin. If u_1, $u_2 : \Omega \to \mathbf{R}$ are solutions of the minimal surface equation with $u_1 \leq u_2$ and $u_1(0) = u_2(0)$, then $u_1 \equiv u_2$.*

Since any smooth hypersurface is locally a graph over a hyperplane, Lemma 1.3 gives a maximum principle for smooth minimal hypersurfaces.

Thus far, the examples of minimal submanifolds have all been smooth. The simplest non-smooth example is given by a pair of planes intersecting transversely along a line. To get an example that is not even immersed, one can take three half–planes meeting along a line with an angle of $2\pi/3$ between each adjacent pair.

2. Monotonicity and the mean value inequality

Monotonicity formulas and mean value inequalities play a fundamental role in many areas of geometric analysis.

Proposition 2.1. *Suppose that $\Sigma^k \subset \mathbf{R}^n$ is a minimal submanifold and $x_0 \in \mathbf{R}^n$; then for all $0 < s < t$*

$$(2.1) \quad t^{-k}\, Vol(B_t(x_0) \cap \Sigma) - s^{-k}\, Vol(B_s(x_0) \cap \Sigma) = \int_{(B_t(x_0) \setminus B_s(x_0)) \cap \Sigma} \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}}\,.$$

Notice that $(x - x_0)^N$ vanishes precisely when Σ is conical about x_0, i.e., when Σ is invariant under dilations about x_0. As a corollary, we get the following:

Corollary 2.2. *Suppose that $\Sigma^k \subset \mathbf{R}^n$ is a minimal submanifold and $x_0 \in \mathbf{R}^n$; then the function*

$$(2.2) \qquad \Theta_{x_0}(s) = \frac{Vol(B_s(x_0) \cap \Sigma)}{Vol(B_s \subset \mathbf{R}^k)}$$

is a nondecreasing function of s. Moreover, $\Theta_{x_0}(s)$ is constant in s if and only if Σ is conical about x_0.

Of course, if x_0 is a smooth point of Σ, then $\lim_{s \to 0} \Theta_{x_0}(s) = 1$. We will later see that the converse is also true; this will be a consequence of the Allard regularity theorem.

The monotonicity of area is a very useful tool in the regularity theory for minimal surfaces — at least when there is some *a priori* area bound. For instance, this monotonicity and a compactness argument allow one to reduce many regularity questions to questions about minimal cones (this was a key observation of W. Fleming in his work on the Bernstein problem; see Section 4).

Arguing as in Proposition 2.1, we get a weighted monotonicity:

Proposition 2.3. *If $\Sigma^k \subset \mathbf{R}^n$ is a minimal submanifold, $x_0 \in \mathbf{R}^n$, and f is a function on Σ, then*

$$(2.3) \qquad t^{-k} \int_{B_t(x_0) \cap \Sigma} f - s^{-k} \int_{B_s(x_0) \cap \Sigma} f$$

$$= \int_{(B_t(x_0) \setminus B_s(x_0)) \cap \Sigma} f \, \frac{|(x - x_0)^N|^2}{|x - x_0|^{k+2}} + \frac{1}{2} \int_s^t \tau^{-k-1} \int_{B_\tau(x_0) \cap \Sigma} \left(\tau^2 - |x - x_0|^2 \right) \Delta_\Sigma f \, d\tau \, .$$

We get immediately the following mean value inequality for the special case of non–negative subharmonic functions:

Corollary 2.4. *Suppose that $\Sigma^k \subset \mathbf{R}^n$ is a minimal submanifold, $x_0 \in \mathbf{R}^n$, and f is a non–negative subharmonic function on Σ; then*

$$(2.4) \qquad s^{-k} \int_{B_s(x_0) \cap \Sigma} f$$

is a nondecreasing function of s. In particular, if $x_0 \in \Sigma$, then for all $s > 0$

$$(2.5) \qquad f(x_0) \leq \frac{\int_{B_s(x_0) \cap \Sigma} f}{Vol(B_s \subset \mathbf{R}^k)} \, .$$

3. Rado's theorem

One of the most basic questions is what does the boundary $\partial \Sigma$ tell us about a compact minimal submanifold Σ? We have already seen that Σ must lie in the convex hull of $\partial \Sigma$, but there are many other theorems of this nature. One of the first is a beautiful result of Rado which says that if $\partial \Sigma$ is a graph over the boundary of a convex set in $\mathbf{R}^2$, then Σ is also graph (and hence embedded). The proof of this uses basic properties of nodal lines for harmonic functions.

Theorem 3.1. *Suppose that $\Omega \subset \mathbf{R}^2$ is a convex subset and $\sigma \subset \mathbf{R}^3$ is a simple closed curve which is graphical over $\partial \Omega$. Then any minimal disk $\Sigma \subset \mathbf{R}^3$ with $\partial \Sigma = \sigma$ must be graphical over Ω and hence unique by the maximum principle.*

Proof. (Sketch.) The proof is by contradiction, so suppose that Σ is such a minimal disk and $x \in \Sigma$ is a point where the tangent plane to Σ is vertical. Consequently, there exists $(a, b) \neq (0, 0)$ such that

$$(3.1) \qquad \nabla_\Sigma (a \, x_1 + b \, x_2)(x) = 0 \, .$$

By Proposition 1.1, $a \, x_1 + b \, x_2$ is harmonic on Σ (since it is a linear combination of coordinate functions). The local structure of nodal sets of harmonic functions (see, e.g., [CM1]) then gives that the level set

$$(3.2) \qquad \{ y \in \Sigma \, | \, a \, x_1 + b \, x_2(y) = a \, x_1 + b \, x_2(x) \}$$

has a singularity at x where at least four different curves meet. If two of these nodal curves were to meet again, then there would be a closed nodal curve which must bound a disk (since Σ is a disk). By the maximum principle, $a \, x_1 + b \, x_2$ would have to be constant on this disk and hence constant on Σ by unique continuation. This would imply that $\sigma = \partial \Sigma$ is contained in the plane given by (3.2). Since this is impossible, we conclude that all of these curves go to the boundary without intersecting again.

In other words, the plane in $\mathbf{R}^3$ given by (3.2) intersects σ in at least four points. However, since $\Omega \subset \mathbf{R}^2$ is convex, $\partial \Omega$ intersects the line given by (3.2) in exactly

two points. Finally, since σ is graphical over $\partial\Omega$, σ intersects the plane in $\mathbf{R}^3$ given by (3.2) in exactly two points, which gives the desired contradiction. $\qquad\square$

4. The theorems of Bernstein and Bers

A classical theorem of S. Bernstein from 1916 says that entire (i.e., defined over all of $\mathbf{R}^2$) minimal graphs are planes. This remarkable theorem of Bernstein was one of the first illustrations of the fact that the solutions to a nonlinear PDE, like the minimal surface equation, can behave quite differently from solutions to a linear equation.

Theorem 4.1. *If $u : \mathbf{R}^2 \to \mathbf{R}$ is an entire solution to the minimal surface equation, then u is an affine function.*

Proof. (Sketch.) We will show that the curvature of the graph vanishes identically; this implies that the unit normal is constant and, hence, the graph must be a plane. The proof follows by combining two facts. First, the area estimate for graphs (1.20) gives

$$(4.1) \qquad\qquad \mathrm{Area}(B_r \cap \mathrm{Graph}_u) \le 2\,\pi\,r^2 \,.$$

This quadratic area growth allows one to construct a sequence of non-negative logarithmic cutoff functions ϕ_j defined on the graph with $\phi_j \to 1$ everywhere and

$$(4.2) \qquad\qquad \lim_{j\to\infty} \int_{\mathrm{Graph}_u} |\nabla\phi_j|^2 = 0 \,.$$

Moreover, since graphs are area-minimizing, they must be stable. We can therefore use ϕ_j in the stability inequality (1.7) to get

$$(4.3) \qquad\qquad \int_{\mathrm{Graph}_u} \phi_j^2\,|A|^2 \le \int_{\mathrm{Graph}_u} |\nabla\phi_j|^2 \,.$$

Combining these gives that $|A|^2$ is zero, as desired. $\qquad\square$

Rather surprisingly, this result very much depended on the dimension. The combined efforts of E. De Giorgi, F. J. Almgren, Jr., and J. Simons finally gave:

Theorem 4.2. *If $u : \mathbf{R}^{n-1} \to \mathbf{R}$ is an entire solution to the minimal surface equation and $n \le 8$, then u is an affine function.*

However, in 1969 E. Bombieri, De Giorgi, and E. Giusti constructed entire non–affine solutions to the minimal surface equation on $\mathbf{R}^8$ and an area–minimizing singular cone in $\mathbf{R}^8$. In fact, they showed that for $m \ge 4$ the cones

$$(4.4) \qquad C_m = \{(x_1,\ldots,x_{2m}) \mid x_1^2 + \cdots + x_m^2 = x_{m+1}^2 + \cdots + x_{2m}^2\} \subset \mathbf{R}^{2m}$$

are area–minimizing (and obviously singular at the origin).

In contrast to the entire case, exterior solutions of the minimal graph equation, i.e., solutions on $\mathbf{R}^2 \setminus B_1$, are much more plentiful. In this case, L. Bers proved that ∇u actually has an asymptotic limit:

Theorem 4.3. *If u is a C^2 solution to the minimal surface equation on $\mathbf{R}^2 \setminus B_1$, then ∇u has a limit at infinity (i.e., there is an asymptotic tangent plane).*

Bers' theorem was extended to higher dimensions by L. Simon:

Theorem 4.4. *If u is a C^2 solution to the minimal surface equation on $\mathbf{R}^n \setminus B_1$, then either*

- *$|\nabla u|$ is bounded and ∇u has a limit at infinity.*
- *All tangent cones at infinity are of the form $\Sigma \times \mathbf{R}$ where Σ is singular.*

Bernstein's theorem has had many other interesting generalizations, some of which will be discussed later.

5. SIMONS INEQUALITY

In this section, we recall a very useful differential inequality for the Laplacian of the norm squared of the second fundamental form of a minimal hypersurface Σ in $\mathbf{R}^n$ and illustrate its role in a priori estimates. This inequality, originally due to J. Simons, is:

Lemma 5.1. *If $\Sigma^{n-1} \subset \mathbf{R}^n$ is a minimal hypersurface, then*

$$(5.1) \qquad \Delta_\Sigma |A|^2 = -2\,|A|^4 + 2|\nabla_\Sigma A|^2 \geq -2\,|A|^4\,.$$

An inequality of the type (5.1) on its own does not lead to pointwise bounds on $|A|^2$ because of the nonlinearity. However, it does lead to estimates if a "scale–invariant energy" is small. For example, H. Choi and Schoen used (5.1) to prove:

Theorem 5.2. *There exists $\epsilon > 0$ so that if $0 \in \Sigma \subset B_r(0)$ with $\partial\Sigma \subset \partial B_r(0)$ is a minimal surface with*

$$(5.2) \qquad \int |A|^2 \leq \epsilon\,,$$

then

$$(5.3) \qquad |A|^2(0) \leq r^{-2}\,.$$

6. HEINZ'S CURVATURE ESTIMATE FOR GRAPHS

One of the key themes in minimal surface theory is the usefulness of a priori estimates. A basic example is the curvature estimate of E. Heinz for graphs. Heinz's estimate gives an effective version of the Bernstein's theorem; namely, letting the radius r_0 go to infinity in (6.1) implies that $|A|$ vanishes, thus giving Bernstein's theorem.

Theorem 6.1. *If $D_{r_0} \subset \mathbf{R}^2$ and $u : D_{r_0} \to \mathbf{R}$ satisfies the minimal surface equation, then for $\Sigma = Graph_u$ and $0 < \sigma \leq r_0$*

$$(6.1) \qquad \sigma^2 \sup_{D_{r_0-\sigma}} |A|^2 \leq C\,.$$

Proof. (Sketch.) Observe first that it suffices to prove the estimate for $\sigma = r_0$, i.e., to show that

$$(6.2) \qquad |A|^2(0, u(0)) \leq C\,r_0^{-2}\,.$$

Recall that minimal graphs are automatically stable. As in the proof of Theorem 4.1, the area estimate for graphs (1.20) allows us to use a logarithmic cutoff function in the the stability inequality (1.7) to get that

$$(6.3) \qquad \int_{B_{r_1} \cap \mathrm{Graph}_u} |A|^2 \leq \frac{C}{\log(r_0/r_1)}\,.$$

Taking r_0/r_1 sufficiently large, we can then apply Theorem 5.2 to get (6.2). $\qquad\square$

7. Embedded minimal disks with area bounds

In the early nineteen–eighties Schoen and Simon extended the theorem of Bernstein to complete simply connected embedded minimal surfaces in $\mathbf{R}^3$ with quadratic area growth. A surface Σ is said to have quadratic area growth if for all $r > 0$, the intersection of the surface with the ball in $\mathbf{R}^3$ of radius r and center at the origin is bounded by $C\,r^2$ for a fixed constant C independent of r.

Theorem 7.1. *Let $0 \in \Sigma^2 \subset B_{r_0} = B_{r_0}(x) \subset \mathbf{R}^3$ be an embedded simply connected minimal surface with $\partial\Sigma \subset \partial B_{r_0}$. If $\mu > 0$ and either*

$$(7.1) \qquad\qquad Area(\Sigma) \leq \mu\,r_0^2 \ \ or \ \ \int_{\Sigma} |A|^2 \leq \mu\,,$$

then for the connected component Σ' of $B_{r_0/2}(x_0) \cap \Sigma$ with $0 \in \Sigma'$ we have

$$(7.2) \qquad\qquad \sup_{\Sigma'} |A|^2 \leq C\,r_0^{-2}$$

for some $C = C(\mu)$.

The result of Schoen–Simon was generalized by Colding–Minicozzi to quadratic area growth for *intrinsic* balls (this generalization played an important role in analyzing the local structure of embedded minimal surfaces):

Theorem 7.2. *Given a constant C_I, there exists C_P so that if $\mathcal{B}_{2r_0} \subset \Sigma \subset \mathbf{R}^3$ is an embedded minimal disk satisfying either*

$$(7.3) \qquad\qquad Area(\mathcal{B}_{2r_0}) \leq C_I\,r_0^2 \ \ or \ \ \int_{\mathcal{B}_{2r_0}} |A|^2 \leq C_I\,,$$

then

$$(7.4) \qquad\qquad \sup_{\mathcal{B}_s} |A|^2 \leq C_P\,s^{-2}\,.$$

As an immediate consequence, letting $r_0 \to \infty$ gives Bernstein-type theorems for embedded simply connected minimal surfaces with either bounded density or finite total curvature. Note that Enneper's surface is simply-connected but neither flat nor embedded; this shows that embeddedness is essential for these estimates. Similarly, the catenoid shows that simply-connected is essential. The catenoid is the minimal surface in $\mathbf{R}^3$ given by

$$(7.5) \qquad\qquad \{(\cosh s\,\cos t, \cosh s\,\sin t, s)\,|\,s,t \in \mathbf{R}\}\,.$$

8. Stable minimal surfaces

It turns out that stable minimal surfaces have a priori estimates. Since minimal graphs are stable, the estimates for stable surfaces can be thought of as generalizations of the earlier estimates for graphs. These estimates have been widely applied and are particularly useful when combined with existence results for stable surfaces (such as the solution of the Plateau problem). The starting point for these estimates is that, as we saw in (1.4), stable minimal surfaces satisfy the stability inequality

$$(8.1) \qquad\qquad \int |A|^2\,\phi^2 \leq \int |\nabla\phi|^2\,.$$

We will mention two such estimates. The first is R. Schoen's curvature estimate for stable surfaces:

Theorem 8.1. *There exists a constant C so that if $\Sigma \subset \mathbf{R}^3$ is an immersed stable minimal surface with trivial normal bundle and $\mathcal{B}_{r_0} \subset \Sigma \setminus \partial\Sigma$, then*

$$(8.2) \qquad \sup_{\mathcal{B}_{r_0-\sigma}} |A|^2 \leq C\,\sigma^{-2}\,.$$

The second is an estimate for the area and total curvature of a stable surface is due to Colding–Minicozzi; for simplicity, we will state only the area estimate:

Theorem 8.2. *If $\Sigma \subset \mathbf{R}^3$ is an immersed stable minimal surface with trivial normal bundle and $\mathcal{B}_{r_0} \subset \Sigma \setminus \partial\Sigma$, then*

$$(8.3) \qquad Area\,(\mathcal{B}_{r_0}) \leq 4\pi\, r_0^2/3\,.$$

As mentioned, we can use (8.3) to bound the energy of a cutoff function in the stability inequality and, thus, bound the total curvature of sub-balls. Combining this with the curvature estimate of Theorem 5.2 gives Theorem 8.1. Note that the bound (8.3) is surprisingly sharp; even when Σ is a plane, the area is πr_0^2.

9. Regularity theory

In this section, we survey some of the key ideas in classical regularity theory, such as the role of monotonicity, scaling, ϵ-regularity theorems (such as Allard's theorem) and tangent cone analysis (such as Almgren's refinement of Federer's dimension reducing). We refer to the book [Mo] for a more detailed overview and a general introduction to geometric measure theory.

The starting point for all of this is the monotonicity of volume for a minimal k–dimensional submanifold Σ. Namely, Corollary 2.2 gives that the density

$$(9.1) \qquad \Theta_{x_0}(s) = \frac{\mathrm{Vol}(B_s(x_0) \cap \Sigma)}{\mathrm{Vol}(B_s \subset \mathbf{R}^k)}$$

is a monotone non-decreasing function of s. Consequently, we can define the density Θ_{x_0} at the point x_0 to be the limit as $s \to 0$ of $\Theta_{x_0}(s)$. It also follows easily from monotonicity that the density is semi–continuous as a function of x_0.

9.1. **ϵ–regularity and the singular set.** An ϵ–regularity theorem is a theorem giving that a weak (or generalized) solution is actually smooth at a point if a scale–invariant energy is small enough there. The standard example is the Allard regularity theorem:

Theorem 9.1. *There exists $\delta(k,n) > 0$ such that if $\Sigma \subset \mathbf{R}^n$ is a k–rectifiable stationary varifold (with density at least one a.e.), $x_0 \in \Sigma$, and*

$$(9.2) \qquad \Theta_{x_0} = \lim_{r \to 0} \frac{Vol\,(B_r(x_0) \cap \Sigma)}{Vol\,(B_r \subset \mathbf{R}^k)} < 1 + \delta\,,$$

then Σ is smooth in a neighborhood of x_0.

Similarly, the small total curvature estimate of Theorem 5.2 may be thought of as an ϵ-regularity theorem; in this case, the scale–invariant energy is $\int |A|^2$.

As an application of the ϵ–regularity theorem, Theorem 9.1, we can define the singular set $\mathcal{S}$ of Σ by

$$(9.3) \qquad \mathcal{S} = \{x \in \Sigma \,|\, \Theta_x \geq 1 + \delta\}\,.$$

It follows immediately from the semi–continuity of the density that $\mathcal{S}$ is closed. In order to bound the size of the singular set (e.g., the Hausdorff measure), one combines the ϵ–regularity with simple covering arguments.

This preliminary analysis of the singular set can be refined by doing a so–called tangent cone analysis.

9.2. **Tangent cone analysis.** It is not hard to see that scaling preserves the space of minimal submanifolds of $\mathbf{R}^n$. Namely, if Σ is minimal, then so is

$$(9.4) \qquad \Sigma_{y,\lambda} = \left\{ y + \lambda^{-1}\left(x - y\right) \,\middle|\, x \in \Sigma \right\}.$$

(To see this, simply note that this scaling multiplies the principal curvatures by λ.) Suppose now that we fix the point y and take a sequence $\lambda_j \to 0$. The monotonicity formula bounds the density of the rescaled solution, allowing us to extract a convergent subsequence and limit. This limit, which is called a *tangent cone* at y, achieves equality in the monotonicity formula and, hence, must be homogeneous (i.e., invariant under dilations about y).

The usefulness of tangent cone analysis in regularity theory is based on two key facts. For simplicity, we illustrate these when $\Sigma \subset \mathbf{R}^n$ is an area minimizing hypersurface. First, if any tangent cone at y is a hyperplane $\mathbf{R}^{n-1}$, then Σ is smooth in a neighborhood of y. This follows easily from the Allard regularity theorem since the density at y of the tangent cone is the same as the density at y of Σ. The second key fact, known as "dimension reducing," is due to Almgren and is a refinement of an argument of Federer. To state this, we first stratify the singular set $\mathcal{S}$ of Σ into subsets

$$(9.5) \qquad \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{n-2}\,,$$

where we define $\mathcal{S}_i$ to be the set of points $y \in \mathcal{S}$ so that any linear space contained in any tangent cone at y has dimension at most i. (Note that $\mathcal{S}_{n-1} = \emptyset$ by Allard's Theorem.) The dimension reducing argument then gives that

$$(9.6) \qquad \dim\left(\mathcal{S}_i\right) \le i\,,$$

where dimension means the Hausdorff dimension. In particular, the solution of the Bernstein problem then gives codimension 7 regularity of Σ, i.e., $\dim\left(\mathcal{S}\right) \le n - 8$.

Part 2. **Constructing minimal surfaces**

Thus far, we have mainly dealt with regularity and a priori estimates but have ignored questions of existence. In this part we survey some of the most useful existence results for minimal surfaces. Section 10 gives an overview of the classical Plateau problem. Section 11 recalls the classical Weierstrass representation, including a few modern applications, and the Kapouleas desingularization method. Section 12 deals with producing area minimizing surfaces and questions of embeddedness. Finally, Section 13 recalls the min–max construction for producing unstable minimal surfaces and, in particular, doing so while controlling the topology and guaranteeing embeddedness.

10. The Plateau Problem

The following fundamental existence problem for minimal surfaces is known as the Plateau problem: Given a closed curve Γ, find a minimal surface with boundary Γ. There are various solutions to this problem depending on the exact definition of a

surface (parameterized disk, integral current, $\mathbf{Z}_2$ current, or rectifiable varifold). We shall consider the version of the Plateau problem for parameterized disks; this was solved independently by J. Douglas and T. Rado. The generalization to Riemannian manifolds is due to C. B. Morrey.

Theorem 10.1. *Let $\Gamma \subset \mathbf{R}^3$ be a piecewise C^1 closed Jordan curve. Then there exists a piecewise C^1 map u from $D \subset \mathbf{R}^2$ to $\mathbf{R}^3$ with $u(\partial D) \subset \Gamma$ such that the image minimizes area among all disks with boundary Γ.*

The solution u to the Plateau problem above can easily be seen to be a branched conformal immersion. R. Osserman proved that u does not have true interior branch points; subsequently, R. Gulliver and W. Alt showed that u cannot have false branch points either.

Furthermore, the solution u is as smooth as the boundary curve, even up to the boundary. A very general version of this boundary regularity was proven by S. Hildebrandt; for the case of surfaces in $\mathbf{R}^3$, recall the following result of J. C. C. Nitsche:

Theorem 10.2. *If Γ is a regular Jordan curve of class $C^{k,\alpha}$ where $k \geq 1$ and $0 < \alpha < 1$, then a solution u of the Plateau problem is $C^{k,\alpha}$ on all of $\bar{D}$.*

11. THE WEIERSTRASS REPRESENTATION

The classical Weierstrass representation (see [Os]) takes holomorphic data (a Riemann surface, a meromorphic function, and a holomorphic one–form) and associates a minimal surface in $\mathbf{R}^3$. To be precise, given a Riemann surface Ω, a meromorphic function g on Ω, and a holomorphic one–form ϕ on Ω, then we get a (branched) conformal minimal immersion $F : \Omega \to \mathbf{R}^3$ by

$$(11.1) \qquad F(z) = \text{Re} \int_{\zeta \in \gamma_{z_0,z}} \left(\frac{1}{2} \left(g^{-1}(\zeta) - g(\zeta) \right), \frac{i}{2} \left(g^{-1}(\zeta) + g(\zeta) \right), 1 \right) \phi(\zeta) \, .$$

Here $z_0 \in \Omega$ is a fixed base point and the integration is along a path $\gamma_{z_0,z}$ from z_0 to z. The choice of z_0 changes F by adding a constant. In general, the map F may depend on the choice of path (and hence may not be well–defined); this is known as "the period problem". However, when g has no zeros or poles and Ω is simply connected, then $F(z)$ does not depend on the choice of path $\gamma_{z_0,z}$.

Two standard constructions of minimal surfaces from Weierstrass data are

$$(11.2) \qquad g(z) = z, \, \phi(z) = dz/z, \, \Omega = \mathbf{C} \setminus \{0\} \text{ giving a catenoid} \, ,$$

$$(11.3) \qquad g(z) = \text{e}^{iz}, \, \phi(z) = dz, \, \Omega = \mathbf{C} \text{ giving a helicoid} \, .$$

The Weierstrass representation is particularly useful for constructing immersed minimal surfaces. Typically, it is rather difficult to prove that the resulting immersion is an embedding (i.e., is 1–1), although there are some interesting cases where this can be done. For the first modern example, D. Hoffman and Meeks proved that the surface constructed by Costa was embedded; this was the first new complete finite topology properly embedded minimal surface discovered since the classical catenoid, helicoid, and plane. This led to the discovery of many more such surfaces (see [Ro] for more discussion).

12. AREA–MINIMIZING SURFACES

Perhaps the most natural way to construct minimal surfaces is to look for ones which minimize area, e.g., with fixed boundary, or in a homotopy class, etc. This has the advantage that often it is possible to show that the resulting surface is embedded. We mention a few results along these lines.

The first embeddedness result, due to Meeks and Yau, shows that if the boundary curve is embedded and lies on the boundary of a smooth mean convex set (and it is null–homotopic in this set), then it bounds an embedded least area disk.

Theorem 12.1. [MeYa1] *Let M^3 be a compact Riemannian three–manifold whose boundary is mean convex and let γ be a simple closed curve in ∂M which is null–homotopic in M; then γ is bounded by a least area disk and any such least area disk is properly embedded.*

Note that some restriction on the boundary curve γ is certainly necessary. For instance, if the boundary curve was knotted (e.g., the trefoil), then it could not be spanned by any embedded disk (minimal or otherwise). Prior to the work of Meeks and Yau, embeddedness was known for extremal boundary curves in $\mathbf{R}^3$ with small total curvature by the work of R. Gulliver and J. Spruck.

If we instead fix a homotopy class of maps, then the two fundamental existence results are due to Sacks–Uhlenbeck and Schoen–Yau (with embeddedness proven by Meeks–Yau and Freedman–Hass–Scott, respectively):

Theorem 12.2. *Given M^3, there exist conformal (stable) minimal immersions $u_1, \ldots, u_m : \mathbf{S}^2 \to M$ which generate $\pi_2(M)$ as a $\mathbf{Z}[\pi_1(M)]$ module. Furthermore,*
- *If $u : \mathbf{S}^2 \to M$ and $[u]_{\pi_2} \neq 0$, then $Area(u) \geq \min_i Area(u_i)$.*
- *Each u_i is either an embedding or a 2–1 map onto an embedded 2–sided $\mathbf{R}P^2$.*

Theorem 12.3. *If Σ^2 is a closed surface with genus $g > 0$ and $i_0 : \Sigma \to M^3$ is an embedding which induces an injective map on π_1, then there is a least area embedding with the same action on π_1.*

13. THE MIN–MAX CONSTRUCTION OF MINIMAL SURFACES

Variational arguments can also be used to construct higher index (i.e., non–minimizing) minimal surfaces using the topology of the space of surfaces. There are two basic approaches:
- Applying Morse theory to the energy functional on the space of maps from a fixed surface Σ to M.
- Doing a min–max argument over families of (topologically non–trivial) sweep–outs of M.

The first approach has the advantage that the topological type of the minimal surface is easily fixed; however, the second approach has been more successful at producing embedded minimal surfaces. We will highlight a few key results below but refer to [CD] for a thorough treatment.

Unfortunately, one cannot directly apply Morse theory to the energy functional on the space of maps from a fixed surface because of a lack of compactness (the Palais–Smale Condition C does not hold). To get around this difficulty, Sacks-Uhlenbeck introduce a family of perturbed energy functionals which do satisfy

Condition C and then obtain minimal surfaces as limits of critical points for the perturbed problems:

Theorem 13.1. *If $\pi_k(M) \neq 0$ for some $k > 1$, then there exists a branched immersed minimal 2–sphere in M (for any metric).*

The basic idea of constructing minimal surfaces via min–max arguments and sweep–outs goes back to Birkhoff, who developed it to construct simple closed geodesics on spheres. In particular, when M is a topological 2–sphere, we can find a 1–parameter family of curves starting and ending at point curves so that the induced map $F : \mathbf{S}^2 \to \mathbf{S}^2$ (see fig. 1) has nonzero degree. The min–max argument produces a nontrivial closed geodesic of length less than or equal to the longest curve in the initial one–parameter family. A curve shortening argument gives that the geodesic obtained in this way is simple.
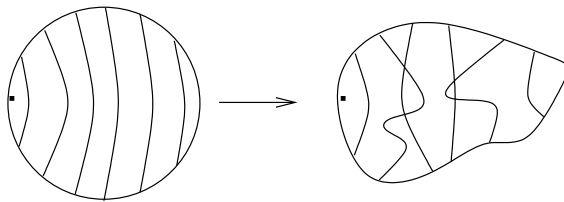

FIGURE 1. A 1–parameter family of curves on a 2–sphere which induces a map $F : \mathbf{S}^2 \to \mathbf{S}^2$ of degree 1. First published in Surveys in Differential Geometry, volume IX, in 2004, published by International Press.

J. Pitts applied a similar argument and geometric measure theory to get that every closed Riemannian three manifold has an embedded minimal surface (his argument was for dimensions up to seven), but he did not estimate the genus of the resulting surface. Finally, F. Smith (under the direction of L. Simon) proved (see [CD]):

Theorem 13.2. *Every metric on a topological 3–sphere M admits an embedded minimal 2–sphere.*

The main new contribution of Smith was to control the topological type of the resulting minimal surface while keeping it embedded.

Part 3. Some applications of minimal surfaces

In this last part, we discuss very briefly a few applications of minimal surfaces. As mentioned in the introduction, there are many to choose from and we have selected just a few.

14.1. The positive mass theorem.
The (Riemannian version of the) positive mass theorem states that an asymptotically flat 3-manifold M with non-negative scalar curvature must have positive mass. The Riemannian manifold M here arises as a maximal space-like slice in a $3+1$-dimensional space-time solution of Einstein's equations.

The asymptotic flatness of M comes from that the space-time models an isolated gravitational system and hence is a perturbation of the vacuum solution outside a large compact set. To make this precise, suppose for simplicity that M has only one end; M is then said to be *asymptotically flat* if there is a compact set $\Omega \subset M$ so that $M \setminus \Omega$ is diffeomorphic to $\mathbf{R}^3 \setminus B_R(0)$ and the metric on $M \setminus \Omega$ can be written as

$$(14.1) \qquad g_{ij} = \left(1 + \frac{\mathcal{M}}{2\,|x|}\right)^4 \delta_{ij} + p_{ij}\,,$$

where

$$(14.2) \qquad |x|^2\,|p_{ij}| + |x|^3\,|D\,p_{ij}| + |x|^4\,|D^2\,p_{ij}| \le C\,.$$

The constant $\mathcal{M}$ is the so called *mass* of M. Observe that the metric g_{ij} is a perturbation of the metric on a constant-time slice in the Schwarzschild space-time of mass $\mathcal{M}$; that is to say, the Schwarzschild metric has $p_{ij} \equiv 0$.

A tensor h is said to be $O(|x|^{-p})$ if $|x|^p\,|h| + |x|^{p+1}\,|D\,h| \le C$. For example, an easy calculation shows that

$$(14.3) \qquad g_{ij} = (1 + 2\,\mathcal{M}/|x|)\,\delta_{ij} + O(|x|^{-2})\,,$$
$$\sqrt{g} \equiv \sqrt{\det g_{ij}} = 1 + 3\,\mathcal{M}\,|x|^{-1} + O(|x|^{-2})\,.$$

The positive mass theorem states that the mass $\mathcal{M}$ of such an M must be non-negative:

Theorem 14.1. [ScYa] *With M as above, $\mathcal{M} \ge 0$.*

There is a rigidity theorem as well which states that the mass vanishes only when M is isometric to $\mathbf{R}^3$:

Theorem 14.2. [ScYa] *If $|\nabla^3 p_{ij}| = O(|x|^{-5})$ and $\mathcal{M} = 0$ in Theorem 14.1, then M is <u>isometric</u> to $\mathbf{R}^3$.*

We will give a very brief overview of the proof of Theorem 14.1, showing in the process where minimal surfaces appear.

Proof. (Sketch.) The argument will by contradiction, so suppose that the mass is negative. It is not hard to prove that the slab between two parallel planes is mean-convex. That is, we have the following:

Lemma 14.3. *If $\mathcal{M} < 0$ and M is asymptotically flat, then there exist $R_0, h > 0$ so that for $r > R_0$ the sets*

$$(14.4) \qquad C_r = \{|x|^2 \le r^2\,, \ -h \le x_3 \le h\}$$

have strictly mean convex boundary.

Since the compact set C_r is mean convex, we can solve the Plateau problem (as in Section 10) to get an area minimizing (and hence stable) surface $\Gamma_r \subset C_r$ with boundary

$$(14.5) \qquad \partial \Gamma_r = \{|x|^2 = r^2\,, \ x_3 = h\}\,.$$

Using the disk $\{|x|^2 \le r^2\,, x_3 = h\}$ as a comparison surface, we get uniform local area bounds for any such Γ_r. Combining these local area bounds with the a priori curvature estimates for minimizing surfaces, we can take a sequence of r's going to

infinity and find a subsequence of Γ_r's that converge to a <u>complete</u> area-minimizing surface

(14.6) $$\Gamma \subset \{-h \leq x_3 \leq h\}\,.$$

Since Γ is pinched between the planes $\{x_3 = \pm h\}$, the estimates for minimizing surfaces implies that (outside a large compact set) Γ is a graph over the plane $\{x_3 = 0\}$ and hence has quadratic area growth and finite total curvature. Moreover, using the form of the metric g_{ij}, we see that $|\nabla u|$ decays like $|x|^{-1}$ and

(14.7) $$\int_{\sigma_s} k_g = (2\,\pi\,s + O(1))\,(s^{-1} + O(s^{-2})) = 2\,\pi + O(s^{-1})\,,$$

where $\sigma_s = \{x_1^2 + x_2^2 = s^2\} \cap \Gamma$ and k_g is the geodesic curvature of σ_s (as a curve in Γ).

To get the contradiction, one combines stability of Γ with the positive scalar curvature of M to see that no such Γ could have existed.[1] Namely, substituting the Gauss equation into the stability inequality[2] gives

(14.8) $$\int_\Gamma (|A|^2/2 + \mathrm{Scal}_M - K_\Sigma)\phi^2 \leq \int_\Gamma |\nabla\phi|^2\,.$$

Since Γ has quadratic area growth, we can choose a sequence of (logarithmic) cutoff functions in (14.8) to get

(14.9) $$0 < \int_\Sigma (|A|^2/2 + \mathrm{Scal}_M) \leq \int_\Sigma K_\Sigma < \infty\,;$$

since K_Σ may not be positive, we also used that Γ has finite total curvature. Moreover, we used that Scal_M is positive outside a compact set to see that the first integral in (14.9) was positive. Finally, substituting (14.9) into the Gauss-Bonnet formula gives that $\int_{\sigma_s} k_g$ is <u>strictly</u> less than 2π for s large, contradicting (14.7). $\square$

14.2. **Black holes.** Another way that minimal surfaces enter into relativity is through black holes. Suppose that we have a three-dimensional time-slice M in a $3 + 1$-dimensional space–time. For simplicity, assume that M is totally geodesic and hence has non-negative scalar curvature. A closed surface Σ in M is said to be trapped if its mean curvature is everywhere negative with respect to its outward normal. Physically, this means that the surface emits an outward shell of light whose surface area is decreasing everywhere on the surface. The existence of a closed trapped surface implies the existence of a black hole in the space-time.

Given a trapped surface, we can look for the outermost trapped surface containing it; this outermost surface is called an apparent horizon. It is not hard to see that an apparent horizon must be a minimal surface and, moreover, a barrier argument shows that it must be stable. Since M has non-negative scalar curvature, stability in turn implies that it must be diffeomorphic to a sphere. See, for instance, [Br] for references to some results on black holes, horizons, etc.

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[1]M was assumed only to have non-negative scalar curvature. However, a "rounding off" argument shows that the metric on M can be perturbed to have positive scalar curvature outside of a compact set and still have negative mass.

[2]This is the stability inequality in a general 3-manifold; see [CM1].

14.3. **Constant mean curvature surfaces.** At least since the time of Plateau, minimal surfaces have been used to model soap films. This is because the mean curvature of the surface models the surface tension and this is essentially the only force acting on a soap film. Soap bubbles, on other hand, enclose a volume and thus the pressure gives a second counterbalancing force. It follows easily that these two forces are in equilibrium when the surface has constant mean curvature.

For the same reason, constant mean curvature surfaces arise in the isoperimetric problem. Namely, a surface that minimizes surface area while enclosing a fixed volume must have constant mean curvature (or "cmc"). It is not hard to see that such an isoperimetric surface in $\mathbf{R}^n$ must be a round sphere. There are two interesting partial converses to this. First, by a theorem of Hopf, any cmc 2-sphere in $\mathbf{R}^3$ must be round. Second, using the maximum principle ("the method of moving planes") Alexandrov showed that any closed embedded cmc hypersurface in $\mathbf{R}^n$ must be a round sphere. It turned out, however, that not every closed immersed cmc surface is round. The first examples were immersed cmc tori constructed by H. Wente. Kapouleas constructed many new examples, including closed higher genus cmc surfaces.

Many of the techniques developed for studying minimal surfaces generalize to general constant mean curvature surfaces.

14.4. **Finite extinction for Ricci flow.** We close this survey with indicating how minimal surfaces can be used to show that on a homotopy 3-sphere the Ricci flow become extinct in finite time (see [CM2], [Pe] for details).

Let M^3 be a smooth closed orientable 3–manifold and let $g(t)$ be a one–parameter family of metrics on M evolving by the Ricci flow, so

$$(14.10) \qquad\qquad \partial_t g = -2 \operatorname{Ric}_{M_t} .$$

In an earlier section, we saw that there is a natural way of constructing minimal surfaces on many 3-manifolds and that comes from the min–max argument where the minimal of all maximal slices of sweep–outs is a minimal surface. The idea is then to look at how the area of this min–max surface changes under the flow. Geometrically the area measures a kind of width of the 3–manifold and as we will see for certain 3–manifolds (those, like the 3–sphere, whose prime decomposition contains no aspherical factors) the area becomes zero in finite time corresponding to that the solution becomes extinct in finite time.

Fix a continuous map $\beta : [0,1] \to C^0 \cap L^2_1(\mathbf{S}^2, M)$ where $\beta(0)$ and $\beta(1)$ are constant maps so that β is in the nontrivial homotopy class $[\beta]$ (such β exists when M is a homotopy 3-sphere). We define the width $W = W(g, [\beta])$ by

$$(14.11) \qquad\qquad W(g) = \min_{\gamma \in [\beta]} \ \max_{s \in [0,1]} \operatorname{Energy}(\gamma(s)) .$$

The next theorem gives an upper bound for the derivative of $W(g(t))$ under the Ricci flow which forces the solution $g(t)$ to become extinct in finite.

Theorem 14.4. *Let M^3 be a homotopy 3-sphere equipped with a Riemannian metric $g = g(0)$. Under the Ricci flow, the width $W(g(t))$ satisfies*

$$(14.12) \qquad\qquad \frac{d}{dt} W(g(t)) \leq -4\pi + \frac{3}{4(t+C)} W(g(t)) ,$$

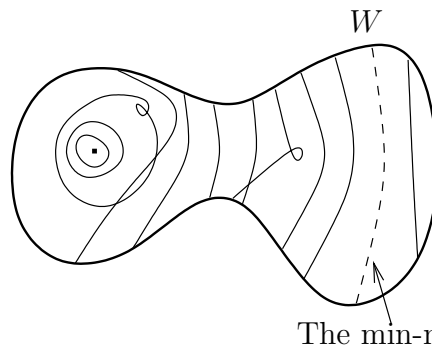*in the sense of the limsup of forward difference quotients. Hence, $g(t)$ must become extinct in finite time.*

FIGURE 2. The sweep–out, the min–max surface, and the width W. First published in the Journal of the American Mathematical Society in 2005, published by the American Mathematical Society.

The 4π in (14.12) comes from the Gauss–Bonnet theorem and the $3/4$ comes from the bound on the minimum of the scalar curvature that the evolution equation implies. Both of these constants matter whereas the constant C depends on the initial metric and the actual value is not important.

To see that (14.12) implies finite extinction time rewrite (14.12) as

$$(14.13) \qquad \frac{d}{dt}\left(W(g(t))\,(t+C)^{-3/4} \right) \le -4\pi\,(t+C)^{-3/4}$$

and integrate to get

$$(14.14) \quad (T+C)^{-3/4}\,W(g(T)) \le C^{-3/4}\,W(g(0)) - 16\,\pi\,\left[(T+C)^{1/4} - C^{1/4} \right].$$

Since $W \ge 0$ by definition and the right hand side of (14.14) would become negative for T sufficiently large we get the claim.

As a corollary of this theorem we get finite extinction time for the Ricci flow.

Corollary 14.5. *Let M^3 be a homotopy 3-sphere equipped with a Riemannian metric $g = g(0)$. Under the Ricci flow $g(t)$ must become extinct in finite time.*

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